Domain representable spaces and topological games

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Domain theory

- K. Martin, "Topological games in domain theory", 2003
- H. Benett, D. Lutzer, "Strong completness Properties in Topology", 2009

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Example

The partially ordered set:

$$P = \{[a,b] : a \leqslant b\}$$

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

The specific relation on P:

$$[a,b] \ll [c,d] \Leftrightarrow [c,d] \subseteq (a,b)$$

The homeomorphism $h\colon \max P \to \mathbb{R}$:

$$h([x,x]) = x$$

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- W. Fleissner, L. Yengulalp, "When $Cp(X)$ is Domain Representable", 2013

κ–domain representable space

We say that a triple $(Q, \ll, B)$ **κ–represents** X and that X is **κ–domain representable** if

(1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a base for $\tau(X)$,

(2) $\ll$ is a transitive, antisymmetric relation on Q,

(3) for all $p, q \in Q$, $p \ll q$ implies $B(p) \supseteq B(q)$,

(4) for all $x \in X$ if $p, p' \in \{q \in Q : x \in B(q)\}$, then there exists $r \in \{q \in Q : x \in B(q)\}$ satisfying $p, q \ll r$,

$(5)_\kappa$ if $D \in [Q]^{<\kappa}$ and $(D, \ll)$ is upward directed (every pair of elements has an upper bound), then $\bigcap\{B(q) : q \in D\} \neq \emptyset$.

If the condition $(5)_\kappa$ is satisfied for every cardinal number κ, we say that a space X is **domain representable**.

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Important results

Theorem[Martin, 2003]

A metric space is a domain representable iff it is completely metrizable.

Theorem[Benett, Lutzer, 2006]

If a space is Čech complete, then it is domain representable.

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Main result

There exists an example of a space, which it is countably $\pi-$ domain representable, but it isn't $\pi-$ domain representable space.

We consider a space

$$\sigma(\{0,1\}^{\omega_1}) = \{x \in \{0,1\}^{\omega_1} : |\alpha < \omega_1 : x(\alpha) = 1| \leqslant \omega\}$$

with the topology generated by the base

$$\mathcal{B} = \{pr_A^{-1}(x) : A \in [\omega_1]^{\leqslant\omega}, x \in \{0,1\}^A\},$$

where $pr_A : \sigma(\{0,1\}^{\omega_1}) \to \{0,1\}^A$ means projection for $A \in [\omega_1]^{\leqslant\omega}$.

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Main result

Let $Q = \mathcal{B}$ and $B : Q \to Q$ be identity.

For every $U \in \mathcal{B}$ let $v(U)$ means a set A such that $pr_A^{-1}(x) = U$ for $x \in \{0,1\}^A$.

We define a relation $\ll$ as follows

$$U \ll V \;\Leftrightarrow\; v(U) \subseteq v(V) \;\Leftrightarrow V \subseteq U$$

for $U, V \in \mathcal{B}$.

The triple $(Q, \ll, B)$ countably π–represents the space $\sigma(\{0,1\}^{\omega_1})$.

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Two players α and β alternately choose open nonempty sets with

β	U_0		U_1			
					$\dots$	
α		V_0		V_1		

Player α wins this play if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise β wins.

Denoted this game by $BM(X)$.

The Banach–Mazur Game

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$$\begin{array}{llllll} \beta & U_0 & & U_1 & & \\ & & & & & \dots \\ \alpha & & V_0 & & V_1 & \end{array}$$

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The strong Choquet game

The *strong Choquet* game is defined as follows:

β	$U_0 \ni x_0$		$U_1 \ni x_1$		
					$\dots$
α		V_0		V_1	

Player α wins if $\bigcap\{V_n : n \in \omega\} \neq \emptyset$. Otherwise β wins.

Denoted this game by $Ch(X)$.

A strategy and a winning strategy

A **strategy** for the player α in the game $BM(X)$ (or $Ch(X)$) is a rule for choosing what to play each round given the full information of moves up until that round.

A **winning strategy** for the player α is a strategy that produces a win for that player α in any game when playing according to that startegy.

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Theorem[Martin, 2003]

If a space X is domain representable, then the player α has a winning strategy in $Ch(X)$.

Theorem[Fleissner, Yengulalp, 2015]

If a space X is countably domain representable, then the player α has a winning strategy in $Ch(X)$.

Theorem[J.B., A. Kucharski]

If the player α has a winning strategy in $Ch(X)$, then X is countably domain representable.

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If the player α has a winning strategy in $Ch(X)$, then X is countably domain representable.

Theorem[J. B., A. Kucharski]

The player α has a winning strategy in the BM(X) iff X is countably π- domain representable.

Thank You for Your attention!